

ALGEBRAIC ORBIFOLD CONFORMAL FIELD THEORIES

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ABSTRACT. We formulate the unitary rational orbifold conformal field theories in the algebraic quantum field theory framework. Under general conditions, we show that the orbifold of a given unitary rational conformal field theories generates a unitary modular category. Many new unitary modular categories are obtained. We also show that the irreducible representations of orbifolds of rank one lattice vertex operator algebras give rise to unitary modular categories and determine the corresponding modular matrices, which has been conjectured for some time.

§1. INTRODUCTION

Cosets and orbifolds are two methods of producing new two dimensional conformal field theories from given ones (cf. [MS]). In [X2, 3, 4, 5], unitary coset conformal field theories are formulated in the algebraic quantum field theory framework and such a formulation is used to solve many questions beyond the reach of other approaches. The main purpose of this paper is to formulate unitary orbifold conformal field theories in the same framework, and to give some applications of this formalism.

There is another approach to conformal field theories by using the theory of vertex operator algebras (cf. [B], [FLM]). In the case of orbifolds this has been studied for example in [DM], [DN]. While there are various advantages to these different approaches, our main results Th. 2.6 and Th. 3.6 have not been obtained previously by other methods.

Under general conditions as specified in Th. 2.6, we show that the orbifold of a given unitary rational conformal field theories generates a unitary modular category. Th. 2.6 gives a large family of new unitary modular categories which can be found in §3.4 and §4. The proof of Th. 2.6 is obtained by using the results of [KLM] and Prop. 2.5. The proof of Prop. 2.5 is based on Galois theory for von Neumann algebras (cf. [ILP] and references therein).

As an application of our general theory, we show in Th. 3.6 that the irreducible representations of orbifolds of rank one lattice vertex operator algebras give rise to unitary modular categories (hence a unitary three dimensional topological quantum field theory, cf. [Tu]) and determine the corresponding modular matrices. More precisely, the simple objects of the modular categories are in one to one correspondence with the irreducible representations of these vertex operator algebras, which were

classified in [DN]. These simple objects and the modular matrices first appeared as examples in [DVVV] based on certain heuristic arguments, and these examples can be clearly interpreted as a conjecture on the existence of certain unitary modular categories with the same modular matrices. Th. 3.6 thus confirms this conjecture.

We will describe the content of this paper in more details. §2.1 and §2.2 are two preliminary sections. We include these sections partly to set up notations. §2.1 is a section on sectors following [L3] and [L4]. In §2.2 the concepts of an irreducible conformal precosheaf \mathcal{A} and its covariant representations as in [GL] are introduced. The definition of modular matrices in [Reh] is given. Note that these modular matrices are very different from the definition of modular matrices in Chap. 13 of [Kac] even though they coincide in all known examples. We shall refer to the modular matrices as defined in §2.2 as *genus 0 modular matrices*. We also note that genus 0 modular matrices determine the fusion rules by definition (cf. (7) of §2.2).

In §2.3 we first define what we call the proper action of a finite group G on \mathcal{A} (Definition 2.1). We show in Prop. 2.1 that if a finite group G acts properly on \mathcal{A} , then the subset of \mathcal{A} which is invariant pointwise under the action of G gives rise to an irreducible conformal precosheaf denoted by \mathcal{A}^G . \mathcal{A}^G is called *the orbifold of \mathcal{A} with respect to G* . Many questions concerning orbifold conformal field theories can be answered by studying covariant representations of \mathcal{A}^G .

In Lemma 2.2 we collect some of the results of [X1] (also cf. [BE1]) which can be applied to our setting. In Lemma 2.3 we show that the proper action as defined in Definition 2.1 is always outer.

In §2.4 we first recall the definition of absolute rationality, or μ -rationality of [KLM]. Prop. 2.4 concerning the calculation of μ -index is essentially Prop. 2.1 of [KLM] (the ideas of the proof appeared in §3 of [X6]). Prop. 2.5 shows that \mathcal{A}_G is always strongly additive if \mathcal{A} is split and strongly additive (cf. §2.4 for the definitions). We use the results of [KLM] and Prop. 2.5 to show in Th. 2.6 that if \mathcal{A} is μ -rational, and G is a finite group acting properly on \mathcal{A} , then \mathcal{A}^G is μ -rational (with a formula for its μ index) and gives rise to a unitary modular category. In Lemma 2.7 we collect some identities from [X4] and [BEK2] which will be used in §3.

In §3.1 we recall from [KM] the branching rules of inclusions

$$Spin(M)_2 \subset SU(M)_1, Spin(M)_2 \subset Spin(M)_1 \times Spin(M)_1 \subset Spin(2M)_1,$$

where $M = 2l$ is an even positive integer, and the numbers in the subscripts are the levels of the positive energy representations of the corresponding loop groups (cf. Chap. 9 of [PS]). By using these branching rules, we show in Lemma 3.1 that $\mathcal{A}_{Spin(M)_2}$ (cf. §2.2 for definitions) is the orbifold of $\mathcal{A}_{SU(M)_1}$ with respect to a natural \mathbb{Z}_2 action (complex conjugation), and in Lemma 3.2 that the coset $\mathcal{A}_{Spin(2M)_1/Spin(M)_2}$ is $\mathcal{A}_{U(1)_{2l}}$.

In §3.2 Prop. 3.3 we calculate the genus 0 modular matrices of $\mathcal{A}_{Spin(M)_2}$ by using Lemma 3.1 and results of [X4] and [BEK2]. One may obtain these results from [TL] but we believe that the ideas in §3.2 may find applications elsewhere. In §3.3 Prop. 3.4 we determine the genus 0 modular matrices of the coset

$\mathcal{A}_{Spin(M)_1 \times Spin(M)_1 / Spin(M)_2}$ by using the method in §4 of [X2] and Prop. 3.3. In §3.4 Lemma 3.5 we show that the coset $\mathcal{A}_{Spin(M)_1 \times Spin(M)_1 / Spin(M)_2}$ is the same as the orbifold of $\mathcal{A}_{U(1)_{2l}}$ with respect to a natural \mathbb{Z}_2 action. By comparing with [DN], the reader may recognize that this \mathbb{Z}_2 action on $\mathcal{A}_{U(1)_{2l}}$ corresponds to the -1 isometry on the rank one lattice vertex operator algebras.

By using Lemma 3.5 and Prop. 3.4, we prove Th. 3.6.

The reader may wonder why we formulate the orbifold of $\mathcal{A}_{U(1)_{2l}}$ with respect to the natural \mathbb{Z}_2 action as a coset instead of considering such an orbifold directly as in [DN]. The reason is that even though one can use the same ideas of §3.2 to set up a system of equations of genus 0 modular matrices for the orbifold, these equations are not sufficient to determine genus 0 modular matrices. One needs to know the conformal dimensions or univalences (cf. §2 of [GL]) of “twisted representations” (cf. The end of §2.4 for the definition). The “twisted representations” in [DN] is defined algebraically. To show that these “twisted representations” in [DN] give rise to covariant representations of our orbifold, one needs to study the analytical properties of the twisted vertex operators in [DN] which is not trivial if one tries this directly.

In §4 we discuss more examples of orbifolds and questions.

§2. ORBIFOLD CFT FROM ALGEBRAIC QFT POINT OF VIEW

§2.1 Sectors. Let M be a properly infinite factor and $\text{End}(M)$ the semigroup of unit preserving endomorphisms of M . In this paper M will always be the unique hyperfinite III_1 factors. Let $\text{Sect}(M)$ denote the quotient of $\text{End}(M)$ modulo unitary equivalence in M . We denote by $[\rho]$ the image of $\rho \in \text{End}(M)$ in $\text{Sect}(M)$.

It follows from [L3] and [L4] that $\text{Sect}(M)$, with M a properly infinite von Neumann algebra, is endowed with a natural involution $\theta \rightarrow \bar{\theta}$; moreover, $\text{Sect}(M)$ is a semiring.

Let $\rho \in \text{End}(M)$ be a normal faithful conditional expectation $\epsilon : M \rightarrow \rho(M)$. We define a number d_ϵ (possibly ∞) by:

$$d_\epsilon^{-2} := \text{Max}\{\lambda \in [0, +\infty) | \epsilon(m_+) \geq \lambda m_+, \forall m_+ \in M_+\}$$

(cf. [PP]).

We define

$$d = \text{Min}_\epsilon \{d_\epsilon | d_\epsilon < \infty\}.$$

d is called the statistical dimension of ρ . It is clear from the definition that the statistical dimension of ρ depends only on the unitary equivalence classes of ρ . The properties of the statistical dimension can be found in [L1], [L3] and [L4].

Denote by $\text{Sect}_0(M)$ those elements of $\text{Sect}(M)$ with finite statistical dimensions. For $\lambda, \mu \in \text{Sect}_0(M)$, let $\text{Hom}(\lambda, \mu)$ denote the space of intertwiners from λ to μ , i.e. $a \in \text{Hom}(\lambda, \mu)$ iff $a\lambda(x) = \mu(x)a$ for any $x \in M$. $\text{Hom}(\lambda, \mu)$ is a finite dimensional vector space and we use $\langle \lambda, \mu \rangle$ to denote the dimension of this space. $\langle \lambda, \mu \rangle$ depends only on $[\lambda]$ and $[\mu]$. Moreover we have $\langle \nu\lambda, \mu \rangle = \langle \lambda, \bar{\nu}\mu \rangle$, $\langle \nu\lambda, \mu \rangle = \langle \nu, \mu\bar{\lambda} \rangle$ which

follows from Frobenius duality (See [L2] or [Y]). We will also use the following notation: if μ is a subsector of λ , we will write as $\mu \prec \lambda$ or $\lambda \succ \mu$. A sector is said to be irreducible if it has only one subsector.

§2.2 The irreducible conformal precosheaf and its representations. In this section we recall the notion of irreducible conformal precosheaf and its covariant representations as described in [GL].

By an *interval* we shall always mean an open connected subset I of S^1 such that I and the interior I' of its complement are non-empty. We shall denote by \mathcal{I} the set of intervals in S^1 . We shall denote by $PSL(2, \mathbf{R})$ the group of conformal transformations on the complex plane that preserve the orientation and leave the unit circle S^1 globally invariant. Denote by \mathbf{G} the universal covering group of $PSL(2, \mathbf{R})$. Notice that \mathbf{G} is a simple Lie group and has a natural action on the unit circle S^1 .

Denote by $R(\vartheta)$ the (lifting to \mathbf{G} of the) rotation by an angle ϑ . This one-parameter subgroup of \mathbf{G} will be referred to as rotation group (denoted by Rot) in the following. We may associate a one-parameter group with any interval I in the following way. Let I_1 be the upper semi-circle, i.e. the interval $\{e^{i\vartheta}, \vartheta \in (0, \pi)\}$. By using the Cayley transform $C : S^1 \rightarrow \mathbf{R} \cup \{\infty\}$ given by $z \rightarrow -i(z-1)(z+1)^{-1}$, we may identify I_1 with the positive real line \mathbf{R}_+ . Then we consider the one-parameter group $\Lambda_{I_1}(s)$ of diffeomorphisms of S^1 such that

$$C\Lambda_{I_1}(s)C^{-1}x = e^s x, \quad s, x \in \mathbf{R}. \quad C\Lambda_{I_1}(s)C^{-1}x = e^s x, \quad s, x \in \mathbf{R}.$$

We also associate with I_1 the reflection r_{I_1} given by

$$r_{I_1}z = \bar{z}$$

where \bar{z} is the complex conjugate of z . It follows from the definition that Λ_{I_1} restricts to an orientation preserving diffeomorphisms of I_1 , r_{I_1} restricts to an orientation reversing diffeomorphism of I_1 onto I'_1 .

Then, if I is an interval and we choose $g \in \mathbf{G}$ such that $I = gI_1$ we may set

$$\Lambda_I = g\Lambda_{I_1}g^{-1}, \quad r_I = gr_{I_1}g^{-1}.$$

Let r be an orientation reversing isometry of S^1 with $r^2 = 1$ (e.g. r_{I_1}). The action of r on $PSL(2, \mathbf{R})$ by conjugation lifts to an action σ_r on \mathbf{G} , therefore we may consider the semidirect product of $\mathbf{G} \times_{\sigma_r} \mathbf{Z}_2$. Since $\mathbf{G} \times_{\sigma_r} \mathbf{Z}_2$ is a covering of the group generated by $PSL(2, \mathbf{R})$ and r , $\mathbf{G} \times_{\sigma_r} \mathbf{Z}_2$ acts on S^1 . We call (anti-)unitary a representation U of $\mathbf{G} \times_{\sigma_r} \mathbf{Z}_2$ by operators on \mathcal{H} such that $U(g)$ is unitary, resp. antiunitary, when g is orientation preserving, resp. orientation reversing.

Now we are ready to define a conformal precosheaf.

An irreducible conformal precosheaf \mathcal{A} of von Neumann algebras on the intervals of S^1 is a map

$$I \rightarrow \mathcal{A}(I)$$

from \mathcal{I} to the von Neumann algebras on a separable Hilbert space \mathcal{H} that verifies the following properties:

A. Isotony. If I_1, I_2 are intervals and $I_1 \subset I_2$, then

$$\mathcal{A}(I_1) \subset \mathcal{A}(I_2).$$

B. Conformal invariance. There is a nontrivial unitary representation U of \mathbf{G} on \mathcal{H} such that

$$U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI), \quad g \in \mathbf{G}, \quad I \in \mathcal{I}.$$

C. Positivity of the energy. The generator of the rotation subgroup $U(R(\vartheta))$ is positive.

D. Locality. If I_0, I are disjoint intervals then $\mathcal{A}(I_0)$ and $\mathcal{A}(I)$ commute.

The lattice symbol \vee will denote ‘the von Neumann algebra generated by’.

E. Existence of the vacuum. There exists a unit vector Ω (vacuum vector) which is $U(\mathbf{G})$ -invariant and cyclic for $\vee_{I \in \mathcal{I}} \mathcal{A}(I)$.

F. Irreducibility. The only $U(\mathbf{G})$ -invariant vectors are the scalar multiples of Ω .

The term irreducibility is due to the fact (cf. Prop. 1.2 of [GL]) that under the assumption of **F** $\vee_{I \in \mathcal{I}} \mathcal{A}(I) = B(\mathcal{H})$.

A covariant *representation* π of \mathcal{A} is a family of representations π_I of the von Neumann algebras $\mathcal{A}(I)$, $I \in \mathcal{I}$, on a separable Hilbert space \mathcal{H}_π and a unitary representation U_π of the covering group \mathbf{G} of $PSL(2, \mathbf{R})$ such that the following properties hold:

$$\begin{aligned} I \subset \bar{I} &\Rightarrow \pi_{\bar{I}}|_{\mathcal{A}(I)} = \pi_I \quad (\text{isotony}) \\ \text{ad}U_\pi(g) \cdot \pi_I &= \pi_{gI} \cdot \text{ad}U(g) \quad (\text{covariance}). \end{aligned}$$

A covariant representation π is called irreducible if $\vee_{I \in \mathcal{I}} \pi(\mathcal{A}(I)) = B(\mathcal{H}_\pi)$. By our definition the irreducible conformal precosheaf is in fact an irreducible representation of itself and we will call this representation the *vacuum representation*.

Let H be a simply connected simply-laced compact Lie group. By Th. 3.2 of [FG], the vacuum positive energy representation of the loop group LH (cf. [PS]) at level k gives rise to an irreducible conformal precosheaf denoted by \mathcal{A}_{H_k} . By Th. 3.3 of [FG], every irreducible positive energy representation of the loop group LH at level k gives rise to an irreducible covariant representation of \mathcal{A}_{H_k} . When $H_k \subset G_l$ is a connected subgroup of a simply connected Lie group, Prop. 2.2 in [X2] gives an irreducible conformal precosheaf which will be denoted by \mathcal{A}_{G_l/H_k} and this is referred to as *the coset conformal precosheaf*. We will see such examples in §3.

Next we will recall some of the results of [Reh] and introduce notations.

Let $\{[\rho_i], i \in I\}$ be a finite set of equivalence classes of irreducible covariant representations of an irreducible conformal precosheaf with finite index. For the definitions of the conjugation and composition of covariant representations, see §4 of [FG] or §2 of [GL].

Suppose this set is closed under conjugation and composition. We will denote the conjugate of $[\rho_i]$ by $[\rho_{\bar{i}}]$ and identity sector by $[1]$ if no confusion arises, and let $N_{ij}^k = \langle [\rho_i][\rho_j], [\rho_k] \rangle$. Here $\langle x, y \rangle$ denotes the dimension of the space of intertwiners from x to y (denoted by $\text{Hom}(x, y)$) for any representations x and y (By Th. 2.3 of [GL] we don't have to distinguish between local and global intertwiners here). We will denote by $\{T_e\}$ a basis of isometries in $\text{Hom}(\rho_k, \rho_i \rho_j)$. The univalence of ρ_i and the statistical dimension of (cf. §2 of [GL]) will be denoted by ω_{ρ_i} and d_{ρ_i} respectively.

Let ϕ_i be the unique minimal left inverse of ρ_i , define:

$$Y_{ij} := d_{\rho_i} d_{\rho_j} \phi_j(\epsilon(\rho_j, \rho_i)^* \epsilon(\rho_i, \rho_j)^*), \quad (0)$$

where $\epsilon(\rho_j, \rho_i)$ is the unitary braiding operator (cf. [GL]).

We list two properties of Y_{ij} (cf. (5.13), (5.14) of [Reh]) which will be used in the following:

$$Y_{ij} = Y_{ji} = Y_{i\bar{j}}^* = Y_{\bar{i}\bar{j}} \quad (1)$$

$$Y_{ij} = \sum_k N_{ij}^k \frac{\omega_i \omega_j}{\omega_k} d_{\rho_k} \quad (2)$$

Define $\tilde{\sigma} := \sum_i d_{\rho_i}^2 \omega_{\rho_i}^{-1}$. If the matrix (Y_{ij}) is invertible, by Proposition on P.351 of [Reh] $\tilde{\sigma}$ satisfies $|\tilde{\sigma}|^2 = \sum_i d_{\rho_i}^2$. Suppose $\tilde{\sigma} = |\tilde{\sigma}| \exp(ix)$, $x \in \mathbb{R}$. Define matrices

$$S := |\tilde{\sigma}|^{-1} Y, T := C \text{Diag}(\omega_{\rho_i}) \quad (3)$$

where $C := \exp(i\frac{x}{3})$. Then these matrices satisfy the algebra:

$$SS^\dagger = TT^\dagger = id, \quad (4)$$

$$TSTST = S, \quad (5)$$

$$S^2 = \hat{C}, T\hat{C} = \hat{C}T = T, \quad (6)$$

where $\hat{C}_{ij} = \delta_{i\bar{j}}$ is the conjugation matrix. Moreover

$$N_{ij}^k = \sum_m \frac{S_{im} S_{jm} S_{km}^*}{S_{1m}}. \quad (7)$$

(7) is known as Verlinde formula.

We will refer the S, T matrices as defined in (3) as **genus 0 modular matrices of \mathcal{A}** since they are constructed from the fusions rules, monodromies and minimal indices which can be thought as genus 0 data associated to a Conformal Field Theory (cf. [MS]).

It follows from (7) and (4) that any irreducible representation of the commutative ring generated by i 's is of the form $i \rightarrow \frac{S_{ij}}{S_{1j}}$.

§2.3. The orbifolds. Let \mathcal{A} be an irreducible conformal precosheaf on a Hilbert space \mathcal{H} and let G be a finite group. Let $V : G \rightarrow U(\mathcal{H})$ be a faithful¹ unitary representation of G on \mathcal{H} .

Definition 2.1. *We say that G acts properly on \mathcal{A} if the following conditions are satisfied:*

- (1) *For each fixed interval I and each $g \in G$, $\alpha_g(a) := V(g)aV(g^*) \in \mathcal{A}(I), \forall a \in \mathcal{A}(I)$;*
- (2) *For each $g \in G$ and $h \in \mathbf{G}$, $[V(g), U(h)] = 0$; moreover $V(g)\Omega = \Omega, \forall g \in G$.*

Suppose that a finite group G acts properly on \mathcal{A} as above. For each interval I , define $\mathcal{B}(I) := \{a \in \mathcal{A}(I) | V(g)aV(g^*) = a, \forall g \in G\}$. Let $\mathcal{H}_0 = \{x \in \mathcal{H} | V(g)x = x, \forall g \in G\}$ and P_0 the projection from \mathcal{H} to \mathcal{H}_0 . Notice that P_0 commutes with every element of $\mathcal{B}(I)$ and $U(g), \forall g \in \mathbf{G}$.

Define $\mathcal{A}^G(I) := \mathcal{B}(I)P_0$ on \mathcal{H}_0 . The unitary representation U of \mathbf{G} on \mathcal{H} restricts to an unitary representation (still denoted by U) of \mathbf{G} on \mathcal{H}_0 . Then:

2.1 Proposition. *The map $I \in \mathcal{I} \rightarrow \mathcal{A}^G(I)$ on \mathcal{H}_0 together with the unitary representation (still denoted by U) of \mathbf{G} on \mathcal{H}_0 is an irreducible conformal precosheaf.*

Proof. We need to check conditions **A** to **F**. **A, B, C, D, F** are immediate consequences of definitions. To check **E**, we have to show that the vacuum vector Ω is cyclic for $\bigvee_{I \in \mathcal{I}} \mathcal{A}^G(I)$ on \mathcal{H}_0 . Fix an interval I and let $a \in \mathcal{A}(I)$. Then by definition $P_0(a\Omega) = \frac{1}{|G|} \sum_g V(g)a\Omega$. Since $V(g)\Omega = \Omega$, we have

$$P_0(a\Omega) = \frac{1}{|G|} \sum_g \alpha_g(a)\Omega,$$

and so $P_0(a\Omega) \in \mathcal{A}^G(I)\Omega$ since

$$\frac{1}{|G|} \sum_g \alpha_g(a) \in \mathcal{A}^G(I).$$

Since $\mathcal{A}(I)\Omega$ is dense in \mathcal{H} by Reeh-Schlieder theorem (cf. Prop. 1.1 of [GL]), it follows that $\mathcal{A}^G(I)\Omega$ is dense in \mathcal{H}_0 , and this shows **E**.

□

The irreducible conformal precosheaf in Prop. 2.1 will be denoted by \mathcal{A}^G and will be called the *orbifold of \mathcal{A}* with respect to G .

The net $\mathcal{B}(I) \subset \mathcal{A}(I)$ is a standard net of inclusions (cf. [LR]) with conditional expectation ϵ defined by

$$\epsilon := \frac{1}{|G|} \sum_g \alpha_g(a), \forall a \in \mathcal{A}(I).$$

¹If $V : G \rightarrow U(\mathcal{H})$ is not faithful, we can take $G' := G/\ker V$ and consider G' instead.

Note that ϵ has finite index. We can therefore apply the theory in [X1] to this setting (also cf. [BE1-2]). Fix an interval I and let $M := \mathcal{A}^G(I)$. Denote by i (resp. λ) the irreducible covariant representations of \mathcal{A}^G (resp. \mathcal{A}) with finite index, and γ the restriction of the vacuum representation of \mathcal{A} to \mathcal{A}^G . Denote by $b_{i\alpha} \in \mathbb{Z}$ the multiplicity of representation λ which appears in the restriction of representation i when restricting from \mathcal{A} to \mathcal{A}^G . $b_{i\alpha}$ is also known as the branching rules. Then there are maps

$$\lambda \rightarrow a_\lambda, i \rightarrow \sigma_i$$

where $a_\lambda, \sigma_i \in \text{End}(M)$ with some remarkable properties. We list some of the properties in the following:

Lemma 2.2. *The map $\lambda \rightarrow a_\lambda$ as a map of sectors is a ring homomorphism. The map $i \rightarrow \sigma_i$ as a map of sectors is a ring isomorphism. Moreover*

$$\langle a_\lambda, a_\mu \rangle = \langle \lambda, \mu \gamma \rangle, b_{i\lambda} = \langle \sigma_i, a_\lambda \rangle.$$

For the proof of Lemma 2.2, we refer the reader to §3 of [X1] or [BE1]. Note that the proof in §3 of [X1] is given in the case of conformal inclusions, but the same proof applies verbatim. We also note that in the proof of above equations (cf. [X1] or [BE1]) strong additivity on $\mathcal{B}(I)$ is assumed to ensure the equivalence of local and global intertwiners. More precisely the equivalence of local and global intertwiners is used in proving Braiding-Fusion equations. The strong additivity assumption is not necessary under the condition of conformal invariance and finite index since under these conditions the equivalence of local and global intertwiners has been proved in Th. 2.3 of [GL].

As noticed in [BE1], if $\lambda = \mu$ is the identity sector, then from above we have: $\langle id, \gamma \rangle = 1$, which implies that the inclusion $\mathcal{B}(I) \subset \mathcal{A}(I)$ is irreducible, i.e., $\mathcal{B}(I)' \cap \mathcal{A}(I) \simeq \mathbb{C}$. We record this result in the following:

Lemma 2.3. *If G acts properly on \mathcal{A} as defined above, then for each I , the action of G on $\mathcal{A}(I)$ is outer, i.e., $\mathcal{B}(I)' \cap \mathcal{A}(I) \simeq \mathbb{C}$ where $\mathcal{B}(I)$ is the fixed point subalgebra of $\mathcal{A}(I)$ under the action of G .*

Since the action of G on $\mathcal{A}(I)$ is outer, there exists a unique normal faithful conditional expectation $\epsilon : \mathcal{A}(I) \rightarrow \mathcal{B}(I)$ given by

$$\epsilon(x) = \frac{1}{|G|} \sum_{k \in K} \alpha_g(x).$$

The index of ϵ is $|G|$ and this is the index of the inclusion $\mathcal{B}(I) \subset \mathcal{A}(I)$. There exists an isometry v in $\mathcal{A}(I)$ (cf. §2 of [LR] or references therein) such that

$$\epsilon(xv^*)v = v^*\tilde{\epsilon}(vx) = \frac{1}{|G|}x, \epsilon(vv^*) = \frac{1}{|G|}, \alpha_g(V)^*\alpha_h(V) = \delta_{g,h}id.$$

Moreover the restriction of the vacuum representation of \mathcal{A} to \mathcal{A}^G , denoted by γ in Lemma 2.2, decomposes as $\gamma = \sum_{\lambda} b_{1\lambda} \lambda$, and $b_{1\lambda} = d_{\lambda}$ where d_{λ} is the statistical dimension of λ . We have

$$\sum_{\lambda} b_{1\lambda} d_{\lambda} = \sum_{\lambda} (d_{\lambda})^2 = |G|.$$

§2.4. Absolute rationality. We first recall some definitions from [KLM]. As in [GL] by an interval of the circle we mean an open connected proper subset of the circle. If I is such an interval then I' will denote the interior of the complement of I in the circle. We will denote by \mathcal{I} the set of such intervals. Let $I_1, I_2 \in \mathcal{I}$. We say that I_1, I_2 are disjoint if $\bar{I}_1 \cap \bar{I}_2 = \emptyset$, where \bar{I} is the closure of I in S^1 . When I_1, I_2 are disjoint, $I_1 \cup I_2$ is called a 1-disconnected interval in [X6]. Denote by \mathcal{I}_2 the set of unions of disjoint 2 elements in \mathcal{I} . Let \mathcal{A} be an irreducible conformal precosheaf as in §2.1. For $E = I_1 \cup I_2 \in \mathcal{I}_2$, let $I_3 \cup I_4$ be the interior of the complement of $I_1 \cup I_2$ in S^1 where I_3, I_4 are disjoint intervals. Let

$$\mathcal{A}(E) := A(I_1) \vee A(I_2), \hat{\mathcal{A}}(E) := (A(I_3) \vee A(I_4))'.$$

Note that $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$. Recall that a net \mathcal{A} is *split* if $\mathcal{A}(I_1) \vee \mathcal{A}(I_2)$ is naturally isomorphic to the tensor product of von Neumann algebras $\mathcal{A}(I_1) \otimes \mathcal{A}(I_2)$ for any disjoint intervals $I_1, I_2 \in \mathcal{I}$. \mathcal{A} is *strongly additive* if $\mathcal{A}(I_1) \vee \mathcal{A}(I_2) = \mathcal{A}(I)$ where $I_1 \cup I_2$ is obtained by removing an interior point from I .

Definition 2.2 (Absolute rationality of [KLM]). \mathcal{A} is said to be *absolute rational*, or μ -rational, if \mathcal{A} is split, strongly additive, and the index $[\hat{\mathcal{A}}(E) : \mathcal{A}(E)]$ is finite for some $E \in \mathcal{I}_2$. The value of the index $[\hat{\mathcal{A}}(E) : \mathcal{A}(E)]$ (it is independent of E by Prop. 5 of [KLM]) is denoted by $\mu_{\mathcal{A}}$ and is called the μ -index of \mathcal{A} . If the index $[\hat{\mathcal{A}}(E) : \mathcal{A}(E)]$ is infinity for some $E \in \mathcal{I}_2$, we define the μ -index of \mathcal{A} to be infinity.

We have:

Proposition 2.4. Suppose $\mathcal{B} \subset \mathcal{A}$ is a standard net of inclusions as defined in 3.1 of [LR]. Let $E \in \mathcal{I}_2$. If $\mathcal{A} \subset \mathcal{B}$ has finite index denoted by $[\mathcal{A} : \mathcal{B}]$ and \mathcal{A} and \mathcal{B} are split, then

$$[\hat{\mathcal{B}}(E) : \mathcal{B}(E)] = [\mathcal{A} : \mathcal{B}]^2 [\hat{\mathcal{A}}(E) : \mathcal{A}(E)].$$

Prop. 2.4 is essentially Prop.21 of [KLM] except that we do not assume that \mathcal{A} and \mathcal{B} are μ -rational. But the proof of Prop.21 of [KLM] applies word by word (also cf. the proof of Th. 3.5 in [X6]).

Proposition 2.5. Let \mathcal{A} be an irreducible conformal precosheaf and let G be a finite group acting properly on \mathcal{A} . Suppose that \mathcal{A} is split and strongly additive. Then \mathcal{A}^G is also split and strongly additive.

Proof. It follows from the definitions that \mathcal{A}^G is split.

Let I be an interval, and I_1, I_2 are the connected components of a set obtained from I by removing an interior point of I . To show \mathcal{A}^G is strongly additive, it is sufficient to show that $\mathcal{B}(I_1) \vee \mathcal{B}(I_2) = \mathcal{B}(I)$.

Let us show that $\mathcal{A}(I_1) \vee \mathcal{B}(I_2) = \mathcal{A}(I)$. First note that $[\mathcal{A}(I) : \mathcal{A}(I_1) \vee \mathcal{B}(I_2)] < \infty$. In fact let $I_2^{(n)} \subset I_2$ be an increasing sequence of intervals such that $I_2^{(n)}$ have one boundary point in common with I_2 , $\bar{I}_1 \cap \bar{I}_2^{(n)} = \emptyset$ and $\cup_n I_2^{(n)} = I_2$. By the additivity of the conformal net \mathcal{A} (cf. §3 of [FJ]), we have that $\mathcal{A}(I_1) \vee \mathcal{B}(I_2^{(n)})$ (resp. $\mathcal{A}(I_1) \vee \mathcal{A}(I_2^{(n)})$) are increasing sequences of von Neumann algebras such that

$$\vee_n \mathcal{A}(I_1) \vee \mathcal{B}(I_2^{(n)}) = \mathcal{A}(I_1) \vee \mathcal{B}(I_2), \vee_n \mathcal{A}(I_1) \vee \mathcal{A}(I_2^{(n)}) = \mathcal{A}(I_1) \vee \mathcal{A}(I_2) = \mathcal{A}(I)$$

where we have used the assumption that \mathcal{A} is strongly additive. By the splitting property we have

$$[\mathcal{A}(I_1) \vee \mathcal{A}(I_2^{(n)}) : \mathcal{A}(I_1) \vee \mathcal{B}(I_2^{(n)})] = [\mathcal{A}(I_1) \otimes \mathcal{A}(I_2^{(n)}) : \mathcal{A}(I_1) \otimes \mathcal{B}(I_2^{(n)})] = |G|.$$

It follows (cf. Prop. 3 of [KLM]) that $[\mathcal{A}(I) : \mathcal{A}(I_1) \vee \mathcal{B}(I_2)] \leq |G|$.

So there exists a faithful normal conditional expectation $\tilde{\epsilon} : \mathcal{A}(I) \rightarrow \mathcal{A}(I_1) \vee \mathcal{B}(I_2)$. Note that

$$\mathcal{B}(I_2) \subset \tilde{\epsilon}(\mathcal{A}(I_2)) \subset \mathcal{A}(I'_1) \cap \mathcal{A}(I) = \mathcal{A}(I_2)$$

and so $\tilde{\epsilon}(\mathcal{A}(I_2))$ is an intermediate von Neumann algebra between $\mathcal{B}(I_2)$ and $\mathcal{A}(I_2)$. So (cf. [ILP] or references therein) there exists a subgroup K of G such that $\tilde{\epsilon}(\mathcal{A}(I_2))$ is the pointwise fixed subalgebra of $\mathcal{A}(I_2)$ under the action of K . Since $\mathcal{B}(I_2) \subset \mathcal{A}(I_2)$ is irreducible by Lemma 2.3, $\tilde{\epsilon}(\mathcal{A}(I_2)) \subset \mathcal{A}(I_2)$ is also irreducible and it follows that there exists a unique conditional expectation from $\mathcal{A}(I_2)$ to $\tilde{\epsilon}(\mathcal{A}(I_2))$ and it is given by

$$\tilde{\epsilon}(x_2) = \frac{1}{|K|} \sum_{k \in K} \alpha_k(x_2).$$

Let us show that K is the trivial subgroup, i.e., if $k \in K$, then k is the identity element of G .

Let $v \in \mathcal{A}(I_2)$ be the isometry as in the end of §2.2 with G replaced by K such that

$$\tilde{\epsilon}(x_2 v^*) v = v^* \tilde{\epsilon}(v x_2) = \frac{1}{|K|} x_2, \epsilon(v v^*) = \frac{1}{|K|}.$$

Define a map $\tilde{\gamma} : \mathcal{A}(I) \rightarrow \mathcal{A}(I_1) \vee \mathcal{B}(I_2)$ by:

$$\tilde{\gamma}(x) := |K| \tilde{\epsilon}(v x v^*), \forall x \in \mathcal{A}(I).$$

One checks easily that

$$\tilde{\gamma}(x_1) = x_1, \tilde{\gamma}(x_2 x'_2) = \tilde{\gamma}(x_2) \tilde{\gamma}(x'_2), \forall x_1 \in \mathcal{A}(I_1), x_2, x'_2 \in \mathcal{A}(I_2).$$

It follows that $\tilde{\gamma}(xy) = \tilde{\gamma}(x)\tilde{\gamma}(y)$ for any $x, y \in \mathcal{A}(I)$ since $A(I)$ is generated by two commuting subalgebras $A(I_1)$ and $A(I_2)$.

For any $k \in K$, define $v_k = \alpha_k(v)$ and

$$\tilde{\alpha}_k(x) = v_k^* \tilde{\gamma}(x) v_k, \forall x \in \mathcal{A}(I).$$

Then one checks that

$$\begin{aligned} \tilde{\alpha}_k(x_1) &= x_1, \tilde{\alpha}_k(x_2 x'_2) = \alpha_k(x_2) \alpha_k(x'_2), \\ \tilde{\alpha}_k(\tilde{\alpha}_{k^{-1}} x_2) &= x_2, \forall x_1 \in \mathcal{A}(I_1), x_2, x'_2 \in \mathcal{A}(I_2). \end{aligned}$$

It follows that

$$\tilde{\alpha}_k(xy) = \tilde{\alpha}_k(x) \tilde{\alpha}_k(y)$$

for any $x, y \in \mathcal{A}(I)$ since $A(I)$ is generated by two commuting subalgebras $A(I_1)$ and $A(I_2)$. One can also check similarly $\tilde{\alpha}_k(\tilde{\alpha}_{k^{-1}}(x)) = x, \forall x \in \mathcal{A}(I)$. So $\tilde{\alpha}_k$ is an automorphism of $\mathcal{A}(I)$. Since $\mathcal{A}(I)$ is a type III factor, there exists a unitary operator $U_k \in B(H_0)$ such that $\tilde{\alpha}_k(x) = U_k x U_k^*, \forall x \in \mathcal{A}(I)$. Since $\tilde{\alpha}_k(x_1) = x_1, \forall x_1 \in \mathcal{A}(I_1)$, we have $U_k \in \mathcal{A}(I_1)' = \mathcal{A}(I_1')$ by Haag duality (cf. §2 of [GL]).

If for all unitary $U' \in \mathcal{A}(I')$, $(U_k U' \Omega, \Omega) = 0$, then $(U' \Omega, U_k \Omega) = 0$, and it follows that $(\mathcal{A}(I') \Omega, U_k^* \Omega) = 0$. Since $\mathcal{A}(I') \Omega$ is dense in \mathcal{H} by Reeh-Schlieder theorem (cf. [GL]), it follows that $U_k^* \Omega = 0$, which implies $U_k^* = 0$ by using Reeh-Schlieder theorem again since $U_k^* \in \mathcal{A}(I_1')$. Hence there exists a unitary $U' \in \mathcal{A}(I')$ such that $(U_k U' \Omega, \Omega) \neq 0$. Note that $\mathcal{A}(I') \subset \mathcal{A}(I_1')$. Replacing U_k by $U_k U'$ if necessary, we may assume that $(U_k \Omega, \Omega) \neq 0$.

Let $g_n \in \mathbf{G}$ be a sequence of elements such that $g_n I_1' = I_1'$ and $g_n I_2$ is an increasing sequence intervals containing I_2 , i.e., $I_2 \subset g_n I_2 \subset g_{n+1} I_2$, and $\cup_n g_n I_2 = I_1'$ (One may take g_n to be a sequence of dilations). Applying $Ad(U(g_n))$ to the equation

$$U_k x_2 = \alpha_k(x_2) U_k$$

and using $\alpha_k(Ad(U(g_n))x_2) = Ad(U(g_n))(\alpha_k(x_2))$, we get

$$Ad(U(g_n))(U_k) Ad(U(g_n))(x_2) = \alpha_k(Ad(U(g_n))x_2) Ad(U(g_n))(U_k).$$

It follow that

$$Ad(U(g_n))(U_k) x_2^{(n)} = \alpha_k(x_2^{(n)}) Ad(U(g_n))(U_k), \forall x_2^{(n)} \in \mathcal{A}(g_n I_2).$$

Let U be a weak limit of $Ad(U(g_n))(U_k)$. Note that

$$U \in \mathcal{A}(I_1')$$

since $Ad(U(g_n))(U_k) \in \mathcal{A}(I_1')$ by our choice of g_n . Since $(Ad(U(g_n))(U_k) \Omega, \Omega) = (U_k \Omega, \Omega) \neq 0$ where we use the fact that Ω is invariant under the action of $U(g_n)$, it follows that

$$(U \Omega, \Omega) = (U_k \Omega, \Omega) \neq 0,$$

so $U \neq 0$, and we have

$$Ux_2^{(n)} = \alpha_k(x_2^{(n)})U, \forall x_2^{(n)} \in \mathcal{A}(g_n I_2).$$

Since $\cup_n g_n I_2 = I'_1$, $\vee_n \mathcal{A}(g_n I_2) = \mathcal{A}(I'_1)$ by the additivity of the conformal net \mathcal{A} (cf. §3 of [FJ]), it follows that

$$U \neq 0, U \in \mathcal{A}(I'_1), Ux = \alpha_k(x)U, \forall x \in \mathcal{A}(I'_1).$$

Recall that α_k is an automorphism of $\mathcal{A}(I'_1)$ and $\mathcal{A}(I'_1)$ is a factor, it follows that

$$UU^* = cid = U^*U, c \neq 0.$$

Change U into $\frac{1}{\sqrt{c}}U$ if necessary, we may assume that U is unitary, and so we have

$$\alpha_k(x) = AdU(x), \forall x \in \mathcal{A}(I'_1).$$

So $U \in B(I'_1)' \cap \mathcal{A}(I'_1)$, and by Lemma 2.3, $AdU(x) = x = \alpha_k(x), \forall x \in \mathcal{A}(I'_1)$. It follow that $V_k x \Omega = x \Omega, \forall x \in \mathcal{A}(I'_1)$, and by Reeh-Schilder theorem $V_k = id$, so k is the identity element in G . Since $k \in K$ is arbitrary, we have shown that K is the trivial group.

So $\mathcal{A}(I_1) \vee \mathcal{B}(I_2) = \mathcal{A}(I)$, and

$$\mathcal{B}(I) = \epsilon(\mathcal{A}(I_1)) = \epsilon(\mathcal{A}(I_1) \vee \mathcal{B}(I_2)) = \mathcal{B}(I_1) \vee \mathcal{B}(I_2).$$

□

Theorem 2.6. *Let \mathcal{A} be an irreducible conformal precosheaf and let G be a finite group acting properly on \mathcal{A} . Suppose that \mathcal{A} is absolutely rational or μ -rational as in definition 2.2. Then:*

- (1): \mathcal{A}^G is absolutely rational or μ -rational and $\mu_{\mathcal{A}^G} = |G|^2 \mu_{\mathcal{A}}$;
- (2): There are only a finite number of irreducible covariant representations of \mathcal{A}^G and they give rise to a unitary modular category as defined in II.5 of [Tu] by the consturction as given in §1.7 of [X5].

Proof:. Ad (1): Note that $\mathcal{B}(I) := \{a \in \mathcal{A}(I) | V(g)aV(g^*) = a, \forall g \in G\} \subset \mathcal{A}(I)$ is a standard net with index $|G|$ by our definitions. By Prop. 2.2, for any $E \in \mathcal{I}_2$,

$$[\hat{\mathcal{A}}^G(E) : \mathcal{A}^G(E)] = |G|^2 [\hat{\mathcal{A}}(E) : \mathcal{A}(E)] = |G|^2 \mu_{\mathcal{A}}.$$

This proves (1).

Ad (2): This follows from Prop. 2.5, Cor. 32 of [KLM] and (2) of Cor. 1.7.3 of [X5].

□

Suppose that \mathcal{A} and G satisfy the assumptions of Th. 2.6. By Th. 2.6 and Th. 30 of [KLM], \mathcal{A}^G has only finite number of irreducible representations λ and

$$\sum_{\lambda} d_{\lambda}^2 = \mu_{\mathcal{A}^G}.$$

The set of such λ 's is closed under conjugation and compositions, and by Cor. 32 of [KLM], the Y -matrix as defined in §2.1 for \mathcal{A}^G is non-degenerate, and we will denote the corresponding genus 0 modular matrices by \dot{S}, \dot{T} . Let \dot{C} (resp. C) be the constant as defined in (3) of §2.1 for \mathcal{A}^G (resp. \mathcal{A}). Denote the genus 0 modular matrices of \mathcal{A} by S, T . Let W be the vector space over \mathbb{C} with basis given the irreducible subsectors of $\sigma_i a_{\lambda}, \forall i, \lambda$.

Lemma 2.7. *Under the conditions above we have:*

- (1): $\sum_{\lambda} b_{i\lambda} \dot{S}_{\lambda\nu} = \sum_k S_{ik} b_{k\nu}$;
- (2): $\dot{C}^3 = C^3$;
- (3): *The dimension of the vector space W is $\sum_{i,\lambda} b_{i\lambda}^2$.*

Proof. (1) and (2) follows from the proof of Prop. 3.1 in [X4] (also cf. the remark after Prop. 3.1 in [X4]). (3) follows from Cor. 4.12 of [BEK2].

□

An irreducible covariant representation λ of \mathcal{A}^G is called an *untwisted* representation if $b_{i\lambda} \neq 0$ for some i . These are representations of \mathcal{A}^G which appear as subrepresentations in the restriction of some representation of \mathcal{A} to \mathcal{A}^G . A representation is called *twisted* if it is not untwisted. Note that $\sum_{\lambda} d_{\lambda} b_{i\lambda} = d_i |G|$, and $b_{1\lambda} = d_{\lambda}$. So we have

$$\sum_{\lambda \text{ untwisted}} d_{\lambda}^2 \leq \sum_i \left(\sum_{\lambda} d_{\lambda} b_{i\lambda} \right)^2 = |G| + \sum_{i \neq 1} d_i^2 |G|^2 < |G|^2 + \sum_{i \neq 1} d_i^2 |G|^2 = \mu_{\mathcal{A}^G}$$

if G is not a trivial group, where in the last = we have used Th. 2.6. It follows that the set of twisted representations of \mathcal{A}^G is not empty. This is very different from the case of cosets, cf. [X4] Cor. 3.2 where it was shown that under certain conditions there are no twisted representations for the coset.

§3. A CLASS OF ORBIFOLDS

§3.1 Some inclusions. We recall from §4 of [KM] the branching rules of inclusions

$$Spin(M)_2 \subset SU(M)_1, Spin(M)_2 \subset Spin(M)_1 \times Spin(M)_1 \subset Spin(2M)_1,$$

where $M = 2l$ is an even positive integer and $l \geq 3$.

We will use H_x to denote the Hilbert space of the positive energy representation of a loop group (cf. §9 of [PS]) with integrable weight x .

For the inclusion $Spin(M)_2 \subset SU(M)_1$, we have

$$\begin{aligned} H_{\tilde{\Lambda}_0} &= H_{2\Lambda_0} + H_{2\Lambda_1}, H_{\tilde{\Lambda}_l} = H_{2\Lambda_l} + H_{2\Lambda_{l-1}}; \\ H_{\tilde{\Lambda}_1} &= H_{\Lambda_0 + \Lambda_1}, H_{\tilde{\Lambda}_{2l-1}} = H_{\Lambda_0 + \Lambda_1}; \\ H_{\tilde{\Lambda}_{l-1}} &= H_{\Lambda_{l-1} + \Lambda_1}, H_{\tilde{\Lambda}_{l+1}} = H_{\Lambda_{l-1} + \Lambda_1}; \\ H_{\tilde{\Lambda}_i} &= H_{\Lambda_i} H_{\tilde{\Lambda}_{2l-i}} = H_{\Lambda_i}, 2 \leq i \leq l-2. \end{aligned} \quad (1)$$

Where $\tilde{\Lambda}_i, 0 \leq i \leq 2l-1$ are the $2l$ integrable weights of $LSU(M)$ at level 1, and

$$2\Lambda_0, 2\Lambda_1, 2\Lambda_l, 2\Lambda_{l-1}, \Lambda_0 + \Lambda_1, \Lambda_{l-1} + \Lambda_1, \Lambda_i, 2 \leq i \leq l-2$$

are integrable weights of $LSpin(M)$ at level 2.

$$H_{\dot{\Lambda}} \otimes H_{\ddot{\Lambda}} = \sum_{\Lambda, \dot{\Lambda} + \ddot{\Lambda} - \Lambda \in Q} H_{(\dot{\Lambda}, \ddot{\Lambda}; \Lambda)} \otimes H_{\Lambda} \quad (2)$$

Where Q is the root lattice of $Spin(M)$ (cf. §1.3 of [KW]), and $(\dot{\Lambda}, \ddot{\Lambda})$ are the integrable weights (both at level 1) of $LSpin(M) \times LSpin(M)$.

The irreducible conformal precosheaf $\mathcal{A}_{U(1)_{2l}}$ associated with $LU(1)$ at level $2l$ is studied in §3.5 of [X5]. We have $\mu_{\mathcal{A}_{U(1)_{2l}}} = 2l$, and there are exactly $2l$ irreducible representations of $\mathcal{A}_{U(1)_{2l}}$ which is labeled by integer $k, 0 \leq k \leq 2l-1$. By identifying $\mathbb{R}^{2M} = (x, y) \rightarrow x + iy \in \mathbb{C}^M$ where x, y are column vectors with M real entries, we have the following natural inclusion $SU(M)_1 \times U(1)_M \subset Spin(2M)_1$ where $U(1)$ acts on \mathbb{C}^M as a complex scalar. We have:

$$H_{\hat{\Lambda}_0} = \sum_{0 \leq i \leq l} H_{\tilde{\Lambda}_{2i}} \otimes H_{2i} \quad (3)$$

where $H_{\hat{\Lambda}_0}$ is the vacuum representation of $LSpin(2M)$ at level 1.

Note that we have natural inclusion $Spin(M)_2 \subset SU(M)_1 \subset Spin(2M)_1$. Define $J := (Id_M, -Id_M) \in SO(2M)$ and lift it to $Spin(2M)$. Note that for $A \in SU(M)$, $JAJ = \bar{A}$, and $JAJ = A$ if $A \in Spin(M)$. The operator J preserves $H_{\hat{\Lambda}_0}$, and one checks

$$J.x = x, \forall x \in H_{\Lambda_0}, J.x = -x, \forall x \in H_{2\Lambda_1}.$$

Moreover on $H_{\hat{\Lambda}_0}$ J commutes with the action of \mathbf{G} . It follows that AdJ generates a proper \mathbb{Z}_2 action on $\mathcal{A}_{SU(M)_1}$, and by the first equation in (1) we have:

Lemma 3.1.

$$\mathcal{A}_{Spin(M)_2} = \mathcal{A}_{SU(M)_1}^{\mathbb{Z}_2}.$$

The next Lemma will be used in §3.4.

Lemma 3.2. $\mathcal{A}_{Spin(2M)_2/Spin(M)_1} = \mathcal{A}_{U(1)_M}$.

Proof. From definition we have

$$\mathcal{A}_{U(1)_M}(I) \subset \mathcal{A}_{Spin(2M)_2/Spin(M)_1}(I).$$

Since the modular automorphism group of $\mathcal{A}_{Spin(2M)_2/Spin(M)_1}(I)$ with respect to the vacuum vector Ω is geometric (cf. §2 of [GL]) and fixes globally $\mathcal{A}_{U(1)_M}(I)$, by Takesaki's theorem (cf. [T]) we just have to show that

$$\overline{\mathcal{A}_{U(1)_M}(I)\Omega} = \overline{\mathcal{A}_{Spin(2M)/Spin(M)}(I)\Omega}.$$

By Reeh-Schlieder's theorem (cf. §2 of [GL]) it is sufficient to show that

$$\overline{LU(1)_M\Omega} = \overline{\bigvee_{I \in \mathcal{I}} \mathcal{A}_{Spin(2M)/Spin(M)}(I)\Omega}.$$

Note that by the branching rules (1) and (3)

$$\overline{\bigvee_{I \in \mathcal{I}} \mathcal{A}_{Spin(2M)/Spin(M)}(I)\Omega} \subset \overline{LU(1)_M\Omega}$$

and the proof is complete.

□

§3.2. Genus 0 modular matrices of $Spin(M)_2$. In this section we determine the genus 0 modular matrices of $\mathcal{A}_{Spin(M)_2}$ by using Lemma 3.1 and Th. 2.6. By Th. 2.6, $\mathcal{A}_{Spin(M)_2}$ is absolute rational with μ -index equal to $4M$. By Cor.9 of [KLM], each irreducible representation of $\mathcal{A}_{Spin(M)_2}$ has finite index. The following is a list of irreducible representations of $\mathcal{A}_{Spin(M)_2}$ (as before we use the integrable weights to denote the corresponding representations):

$$\begin{aligned} \widehat{1} &:= 2\Lambda_0, \widehat{j} := 2\Lambda_1, \widehat{\phi}_l^1 := 2\Lambda_{l-1}, \widehat{\phi}_l^2 := 2\Lambda_l; \\ \widehat{\phi}_1 &:= \Lambda_0 + \Lambda_1, \widehat{\phi}_{l-1} := \Lambda_{l-1} + \Lambda_1, \widehat{\phi}_2 := \Lambda_2, \dots, \widehat{\phi}_{l-2} := \Lambda_{l-2}; \\ \widehat{\sigma}_1 &:= \Lambda_0 + \Lambda_{l-1}, \widehat{\sigma}_2 := \Lambda_0 + \Lambda_1, \widehat{\tau}_2 := \Lambda_1 + \Lambda_{l-1}, \widehat{\tau}_1 := \Lambda_1 + \Lambda_l \end{aligned} \quad (4)$$

where we have chosen our notations so that one may easily compare with the notations of [DVVV]. We will show that the above is in fact a complete list of irreducible representations of $\mathcal{A}_{Spin(M)_2}$.

The center Z of $Spin(M)$ is \mathbb{Z}_4 when l is odd, and $\mathbb{Z}_2 \times \mathbb{Z}_2$ when l is even, and Z acts transitively (cf. Chap. 9 of [PS]) on the set $2\Lambda_0, 2\Lambda_1, 2\Lambda_{l-1}, 2\Lambda_l$. The action is also known as diagram automorphisms as defined in §3.3. Choose a path (localized on an interval I) $P : [0, 2\pi] \rightarrow H$ where $H \subset Spin(M)$ is the Cartan subgroup of $Spin(M)$, such that $P(0) = id$ of $Spin(M)$, and $P(2\pi) \in Z$. AdP gives an outer action of $LSpin(M)$ if $P(2\pi) \neq id$. One has (cf. Chap. 9 of [PS]):

$$\pi_\Lambda(AdP.LSpin(M)) \simeq \pi_{P(0).\Lambda}(LSpin(M)),$$

where $P(0).\Lambda$ denotes the image of the action of $P(0)$ on Λ . From this one can determine the following fusion rules (See §3.3 for the definition of diagram automorphisms):

$$\begin{aligned}\widehat{\phi_l^1}^2 &= \widehat{j}, \widehat{\phi_l^1}^3 = \widehat{\phi_l^2}, \widehat{\phi_l^1}^4 = \widehat{1}, \widehat{\phi_l^1} \widehat{\sigma_1} = \widehat{\tau_2}, \text{ if } l \in 2\mathbb{Z} + 1; \\ \widehat{\phi_l^1}^2 &= \widehat{\phi_l^2}^2 = \widehat{j}^2 = \widehat{1}, \widehat{\phi_l^1} \widehat{\sigma_1} = \widehat{\sigma_1}, \widehat{\phi_l^2} \widehat{\sigma_2} = \widehat{\sigma_2}, \widehat{j} \widehat{\sigma_k} = \widehat{\tau_k}, k = 1, 2, \text{ if } l \in 2\mathbb{Z}.\end{aligned}\tag{5}$$

We will denote by \dot{S}, \dot{T} (resp. S, T) the genus 0 modular matrices of $\mathcal{A}_{Spin(M)_2}$ (resp. $\mathcal{A}_{SU(M)_1}$). Note that by [W2] $S_{\Lambda_k, \Lambda_{k'}} = \frac{1}{\sqrt{2l}} \exp(\frac{2\pi i k k'}{2l}), 0 \leq k, k' \leq 2l - 1$.

Consider the space W as defined before Lemma 2.7. We have $\dim W = \sum_{i, \alpha} b_{i\alpha}^2 = M + 2$ by (3) of Lemma 2.7 and (1) of §3.1. Using (1) of §3.1, Lemma 2.2 one obtains the following equations:

$$\begin{aligned}a_{2\Lambda_0} &= a_{2\Lambda_1} = a_{2\Lambda_{l-1}} = a_{2\Lambda_l} = id, \\ a_{\Lambda_0 + \Lambda_1} &= \sigma_{\tilde{\Lambda}_1} + \sigma_{\tilde{\Lambda}_{2l-1}}, a_{\Lambda_i} = \sigma_{\tilde{\Lambda}_i} + \sigma_{\tilde{\Lambda}_{2l-i}}, 2 \leq i \leq l - 2, \\ a_{\Lambda_{l-1} + \Lambda_l} &= \sigma_{\tilde{\Lambda}_{l-1}} + \sigma_{\tilde{\Lambda}_{l+1}}.\end{aligned}$$

Let W_0 be the subspace with basis $\sigma_{\tilde{\Lambda}_k}, 0 \leq k \leq 2l - 1$. Assume that $W = W_0 \oplus W_1$. Then W_1 has dimension 2.

Let us show that for any irreducible twisted representation x of $\mathcal{A}_{Spin(M)_2}$ (these are representations which do not appear in (1) of §3.1), $\widehat{j}x \neq x$. Since by (1) of Lemma 2.7 and (1) of §3.1 we have

$$\dot{S}_{2\Lambda_0}x + \dot{S}_{2\Lambda_1}x = 0,$$

so by (2) of §2.2

$$Y_{2\Lambda_1}x = \omega_{2\Lambda_1}x / \omega_x = -1.$$

This implies that $2\Lambda_1x = \widehat{j}x \neq x$. Since $\dim W_1 = 2$, and

$$\langle a_x, a_y \rangle = \langle x, y \rangle + \langle x, \widehat{j}y \rangle$$

by Lemma 2.2, it follows that there are exactly 4 twisted representations to generate two dimensional W_1 . But there are exactly 4 twisted representations in (4), so these are all the irreducible representations.

The univalences of the representations in (4) are given by (cf. §1.4 of [KW]):

$$\begin{aligned}\omega_{\widehat{1}} &= \omega_{\widehat{j}} = 1, \omega_{\widehat{\phi_k}} = \exp(\frac{\pi i k(l-k)}{4l}), 1 \leq k \leq l-1, \\ \omega_{\widehat{\sigma_1}} &= \omega_{\widehat{\sigma_2}} = \exp(\frac{\pi i(2l-1)}{8}), \omega_{\widehat{\tau_1}} = \omega_{\widehat{\tau_2}} = -\exp(\frac{\pi i(2l-1)}{8}).\end{aligned}$$

$\sqrt{8l} \times \dot{S}$	$\widehat{1}$	\widehat{j}	$\widehat{\phi_l^j}$	$\widehat{\phi_{k'}}$	$\widehat{\sigma_j}$	$\widehat{\tau_j}$
$\widehat{1}$	1	1	1	2	\sqrt{l}	\sqrt{l}
\widehat{j}	1	1	1	2	$-\sqrt{l}$	$-\sqrt{l}$
$\widehat{\phi_l^i}$	1	1	$(-1)^l$	$2(-1)^{k'}$	b_{ij}	b_{ij}
$\widehat{\phi_k}$	2	2	$2(-1)^k$	$4 \cos \frac{\pi k k'}{2l}$	0	0
$\widehat{\sigma_i}$	\sqrt{l}	$-\sqrt{l}$	b_{ij}	0	a_{ij}	$-a_{ij}$
$\widehat{\tau_i}$	\sqrt{l}	$-\sqrt{l}$	b_{ij}	0	$-a_{ij}$	a_{ij}

Table 3.2. $\sqrt{8l} \times \dot{S}$ -matrix. Here $a_{ij} = \sqrt{\frac{l}{2}}(1 + (2\delta_{i,j} - 1)\exp(-\frac{\pi il}{2}))$, $b_{ij} = (-1)^{l+\delta_{i,j}}\sqrt{l}\exp\frac{\pi il}{2}$, $\delta_{i,j}$ is the usual Delta function, and $1 \leq i, j \leq 2$.

Using (2) of Lemma 2.7 the \dot{T} matrix can be chosen to be

$$T_{xy} = \delta_{x,y} \omega_x \exp\left(\frac{-\pi i(2l-1)}{12}\right) \quad (6)$$

where ω_x is given as above. The \dot{S} -matrix is obtained by using (1) to (7) of §2.1 and Lemma 2.7, and it is given in table 3.2.

Let us explain in more details how table 3.2 is obtained. First note that if β is a twisted representation, from (1) of Lemma 2.7 and (1) we have

$$\dot{S}_{\widehat{\phi_l^1}\beta} = -\dot{S}_{\widehat{\phi_l^2}\beta}, \dot{S}_{\widehat{1}\beta} = -\dot{S}_{\widehat{j}\beta}, \dot{S}_{\widehat{\phi_k}\beta} = 0, 1 \leq k \leq l-1.$$

Turning to nontwisted representations one obtains directly from (1) of Lemma 2.7 that

$$\dot{S}_{\widehat{\phi_k}\widehat{\phi_{k'}}} = 4 \cos\left(\frac{\pi k k'}{l}\right), 1 \leq k, k' \leq l-1.$$

By (1) of Lemma 2.7 we also have:

$$\begin{aligned} \dot{S}_{\widehat{\phi_l^1}\widehat{\phi_k}} + \dot{S}_{\widehat{\phi_l^2}\widehat{\phi_k}} &= \frac{(-1)^{k'}}{l}, \dot{S}_{\widehat{\phi_l^1}\widehat{\phi_l^2}} + \dot{S}_{\widehat{\phi_l^1}\widehat{\phi_l^1}} = \frac{(-1)^l}{2l}; \\ \dot{S}_{\widehat{\phi_l^1}\widehat{j}} + \dot{S}_{\widehat{\phi_l^2}\widehat{j}} &= \frac{1}{2l}, \dot{S}_{\widehat{j}\widehat{\phi_k}} + \dot{S}_{\widehat{1}\widehat{\phi_k}} = \frac{1}{2l}. \end{aligned}$$

These equations and (4) are sufficient to determine the rows labeled by

$$\widehat{1}, \widehat{j}, \widehat{\phi_k}, \widehat{\phi_l^i}, 1 \leq k \leq l-1, i = 1, 2$$

in Table 3.2. By the orthogonality of \dot{S} matrix, the $\widehat{\phi_l^1}$ -th row is orthogonal to the $\widehat{1}$ -th row, and we get the following equation $\dot{S}_{\widehat{\phi_l^1}\widehat{\sigma_1}} = -\dot{S}_{\widehat{\phi_l^1}\widehat{\sigma_2}}$. This and (5) of §2.1 are sufficient to determine all the entries in table 3.2.

One checks easily that the genus 0 modular matrices as given above coincide with the the modular matrices as given in Chap. 13 of [Kac].

Let us record the above results in the following:

Proposition 3.3. *All the irreducible covariant representations of $\mathcal{A}_{Spin(M)_2}$ are listed in (4). The genus 0 modular matrices of $\mathcal{A}_{Spin(M)_2}$ are given by (6) and table 3.2.*

§3.3. The diagonal coset $Spin(M)_2 \subset Spin(M)_1 \times Spin(M)_1$. In this section we determine the modular matrices for diagonal coset $Spin(M)_2 \subset Spin(M)_1 \times Spin(M)_1$. The diagonal cosets in the type A case have been discussed in §4.3 of [X2]. We will use the same idea in §4.3 of [X2] to obtain the the modular matrices for diagonal coset $Spin(M)_2 \subset Spin(M)_1 \times Spin(M)_1$.

First we note that Th. 2.3 of [X2] holds in our case: the only change one needs to make in the proof of Th. 2.3 of [X2] is to replace fermions by real fermions as in [Bo]. Since $Spin(M)_2 \subset Spin(2M)_1$ is cofinite by Lemma 3.2, and $Spin(M)_1 \times Spin(M)_1 \subset Spin(2M)_1$, by Prop. 3.1 of [X2] $Spin(M)_2 \subset Spin(M)_1 \times Spin(M)_1$ is also cofinite.

Recall the set of integrable weights of irreducible positive energy representations of $LSpin(2l)$ at level k is given by:

$$P_+^{(k)} = \{\lambda = \lambda_0\Lambda_0 + \lambda_1\Lambda_1 + \dots\lambda_{l-1}\Lambda_{l-1} + \lambda_l\Lambda_l \mid \lambda_i \in \mathbb{N}, \\ \lambda_0 + \lambda_1 + 2(\lambda_2 + \dots + \lambda_{l-2}) + \lambda_{l-1} + \lambda_l = k\}$$

When l is even, this set admits a $\mathbb{Z}_2 \times \mathbb{Z}_2$ automorphism generated by A_s, A_v where A_s, A_v are given by:

$$A_s(\lambda_0\Lambda_0 + \lambda_1\Lambda_1 + \dots\lambda_{l-1}\Lambda_{l-1} + \lambda_l\Lambda_l) = \lambda_0\Lambda_l + \lambda_1\Lambda_{l-1} + \dots\lambda_{l-1}\Lambda_1 + \lambda_l\Lambda_0, \\ A_v(\lambda_0\Lambda_0 + \lambda_1\Lambda_1 + \dots\lambda_{l-1}\Lambda_{l-1} + \lambda_l\Lambda_l) = \lambda_0\Lambda_1 + \lambda_1\Lambda_0 + \lambda_2\Lambda_2 + \dots\lambda_{l-2}\Lambda_{l-2} + \\ \lambda_{l-1}\Lambda_l + \lambda_l\Lambda_{l-1}.$$

When l is odd, this set admits a \mathbb{Z}_4 automorphism generated by A_s where A_s is given by

$$A_s(\lambda_0\Lambda_0 + \lambda_1\Lambda_1 + \dots\lambda_{l-1}\Lambda_{l-1} + \lambda_l\Lambda_l) = \lambda_0\Lambda_l + \lambda_1\Lambda_{l-1} + \lambda_2\Lambda_{l-2} + \dots\lambda_{l-2}\Lambda_2 + \\ \lambda_{l-1}\Lambda_0 + \lambda_l\Lambda_1.$$

These automorphisms will be called *diagram automorphisms*. Note that the set of diagram automorphisms is $\mathbb{Z}_2 \times \mathbb{Z}_2$ when l is even, and \mathbb{Z}_4 when l is odd.

Th. 4.3 of [X2] now holds for our diagonal coset, with the action of \mathbb{Z}_N there replaced by $\mathbb{Z}_2 \times \mathbb{Z}_2$ when l is even, and \mathbb{Z}_4 when l is odd, since the proof of [X2] applies verbatim. As in §4.3 of [X2], we denote by $[\dot{\Lambda}, \ddot{\Lambda}; \Lambda]$ the orbit of $(\dot{\Lambda}, \ddot{\Lambda}; \Lambda)$

under the diagonal action of the diagram automorphisms. The the following is a list of irreducible representations of $\mathcal{A}_{Spin(M)_1 \times Spin(M)_1 / Spin(M)_2}$.

$$\begin{aligned}
1 &:= [\dot{\Lambda}_0, \ddot{\Lambda}_0; \Lambda_0], j := [\dot{\Lambda}_0, \ddot{\Lambda}_0; 2\Lambda_1], \\
\phi_l^1 &:= [\dot{\Lambda}_0, \ddot{\Lambda}_0; 2\Lambda_{l-1}], \phi_l^2 := [\dot{\Lambda}_0, \ddot{\Lambda}_0; 2\Lambda_l], \text{ if } l \in 2\mathbb{Z}, \\
\phi_l^1 &:= [\dot{\Lambda}_0, \ddot{\Lambda}_1; 2\Lambda_{l-1}], \phi_l^2 := [\dot{\Lambda}_0, \ddot{\Lambda}_1; 2\Lambda_l], \text{ if } l \in 2\mathbb{Z} + 1, \\
\phi_1 &:= [\dot{\Lambda}_0, \ddot{\Lambda}_1; \Lambda_0 + \Lambda_1], \phi_2 := [\dot{\Lambda}_0, \ddot{\Lambda}_0; \Lambda_2], \phi_3 := [\dot{\Lambda}_0, \ddot{\Lambda}_1; \Lambda_3], \dots, \\
\phi_{l-2} &:= [\dot{\Lambda}_0, \ddot{\Lambda}_0; \Lambda_{l-2}], \text{ if } l \in 2\mathbb{Z}, \phi_{l-2} := [\dot{\Lambda}_0, \ddot{\Lambda}_1; \Lambda_{l-2}], \text{ if } l \in 2\mathbb{Z} + 1, \\
\phi_{l-1} &:= [\dot{\Lambda}_0, \ddot{\Lambda}_1; \Lambda_{l-1} + \Lambda_l], \text{ if } l \in 2\mathbb{Z}, \phi_{l-1} := [\dot{\Lambda}_0, \ddot{\Lambda}_0; \Lambda_{l-1} + \Lambda_l], \text{ if } l \in 2\mathbb{Z} + 1, \\
\sigma_1 &:= [\dot{\Lambda}_0, \ddot{\Lambda}_{l-1}; \Lambda_0 + \Lambda_{l-1}], \tau_1 := [\dot{\Lambda}_0, \ddot{\Lambda}_{l-1}; \Lambda_1 + \Lambda_l], \\
\sigma_2 &:= [\dot{\Lambda}_0, \ddot{\Lambda}_l; \Lambda_0 + \Lambda_l], \tau_2 := [\dot{\Lambda}_0, \ddot{\Lambda}_l; \Lambda_1 + \Lambda_{l-1}]
\end{aligned} \tag{7}$$

The above is in fact a complete list of all the irreducible covariant representations of $\mathcal{A}_{Spin(M)_1 \times Spin(M)_1 / Spin(M)_2}$ by Cor. 3.2 of [X4], since the proof of Cor. 3.2 of [X4] applies verbatim in the present case. We have also chosen our notations to make the comparisons with the notations of [DVVV] easy (our l corresponds to N on P. 517, P. 518 of [DVVV]).

The univalences of the above representations are given by:

$$\begin{aligned}
\omega_1 = \omega_j = 1, \omega_{\phi_k} &= \exp\left(\frac{\pi i k^2}{4l}\right), 1 \leq k \leq l-1, \\
\omega_{\sigma_1} = \omega_{\sigma_2} &= \exp\left(\frac{\pi i}{8}\right), \omega_{\tau_1} = \omega_{\tau_2} = -\exp\left(\frac{\pi i}{8}\right)
\end{aligned}$$

Using the remark after Prop. 3.1 of [X4] the T matrix can be chosen to be

$$T_{xy} = \delta_{x,y} \omega_x \exp\left(\frac{-\pi i}{12}\right) \tag{8}$$

where ω_x is given as above. By Prop. 3.4 and (7), one can easily determine the S -matrix (cf. (2) of Lemma 2.2 in [X3]) for $\mathcal{A}_{Spin(M)_1 \times Spin(M)_1 / Spin(M)_2}$. We have chosen our notations in (7) so that the S -matrix is given by table 3.2 with all the hats removed. We record the results of this section in the following:

Proposition 3.4. *All the irreducible covariant representations of*

$$\mathcal{A}_{Spin(M)_1 \times Spin(M)_1 / Spin(M)_2}$$

are given in (7), and its modular matrices are given by (8) and table 3.2 (with all the hats removed).

§3.4. The \mathbb{Z}_2 orbifold. Note that the \mathbb{Z}_2 action on $\mathcal{A}_{U(1)_{2l}}$ given by AdJ as defined before Lemma 3.1 is a proper action on $\mathcal{A}_{U(1)_{2l}}$. The reader familiar with [FLM] may notice that the action given by AdJ corresponds to -1 isometry of rank one lattice vertex operator algebras (cf. [FLM] and [DN]). We have:

Lemma 3.5.

$$\mathcal{A}_{Spin(M)_1 \times Spin(M)_1 / Spin(M)_2} = \mathcal{A}_{U(1)_{2l}}^{\mathbb{Z}_2}.$$

Proof. By definition we have a natural inclusion

$$\mathcal{A}_{Spin(M)_1 \times Spin(M)_1 / Spin(M)_2}(I) \subset \mathcal{A}_{U(1)_{2l}}^{\mathbb{Z}_2}(I), \forall I \in \mathcal{I}$$

since $\mathcal{A}_{Spin(M)_1 \times Spin(M)_1}(I)$ is AdJ invariant. Note that the inclusion has finite index d since $Spin(M)_1 \times Spin(M)_1 \subset Spin(2M)_1$ is a conformal inclusion which has finite index by [Bo]. By Prop. 2.4 we have

$$\mu_{\mathcal{A}_{Spin(M)_1 \times Spin(M)_1 / Spin(M)_2}} \times d^2 = \mu_{\mathcal{A}_{U(1)_{2l}}^{\mathbb{Z}_2}}.$$

But $\mu_{\mathcal{A}_{Spin(M)_1 \times Spin(M)_1 / Spin(M)_2}} = 8l$ by Prop. 3.4, $\mu_{\mathcal{A}_{U(1)_{2l}}^{\mathbb{Z}_2}} = 8l$ by Th. 2.6, it follows that $d^2 = 1$, and so $\mathcal{A} = \mathcal{A}_{U(1)_{2l}}^{\mathbb{Z}_2}$.

□

By Lemma 3.4, Th. 2.6 and Prop. 3.4, we have proved the following theorem:

Theorem 3.6. *All the irreducible representations of $\mathcal{A}_{U(1)_{2l}}^{\mathbb{Z}_2}$ are given by (7) for $l \geq 3$. These irreducible representations give rise to a unitary modular category whose genus 0 modular matrices are given by (8) and table 3.2 with all the hats removed.*

When $l = 2$, $Spin(4) = SU(2) \times SU(2)$, one checks that Th. 3.6 still holds in this case, where the integrable weights of $LSpin(4)$ should be replaced by the integrable weights of $LSU(2) \times LSU(2)$. When $l = 1$, Using the fact that $\mathcal{A}_{U(1)_2} = \mathcal{A}_{SU(2)_1}$ one can check easily that

$$\mathcal{A}_{U(1)_2}^{\mathbb{Z}_2} = \mathcal{A}_{U(1)_8},$$

and $\mathcal{A}_{U(1)_8}$ has already been studied in §3.5 of [X5]. As noted before Th. 3.6, the \mathbb{Z}_2 action on $\mathcal{A}_{U(1)_{2l}}$ given by AdJ corresponding to -1 isometry of rank one lattice vertex operator algebras (cf. [FLM] and [DN]). The classification of irreducible representations of the orbifold rank one lattice vertex operator algebras is given in [DN] which corresponds to the first part of Th. 3.6. We note that the S matrix can be identified with the S matrix on P. 517 and P. 518 of [DVVV]. However there are mistakes in the S matrix on P. 517 and P. 518 of [DVVV] corresponding to the entries of a_{ij}, b_{ij} in Table 3.2. Table 3.2 gives the correct S matrix.

§4. MORE EXAMPLES AND QUESTIONS.

The lattice vertex operator algebras and their automorphism groups provide a rich source of examples of orbifolds (cf. [FLM], [DN], [DGR]). We have determined the genus 0 modular matrices for the orbifold of rank 1 lattice VOAs in Th. 3.6. It will be interesting to generalize this to higher rank cases.

Let $\mathcal{A}_{SU(N)_k}$ be the irreducible conformal precosheaf and let G be a finite subgroup of $SU(N)$. Then there is a natural action of G on $\mathcal{A}_{SU(N)_k}$ and it is easy to check that this action is proper (If the action of G is not faithful, one can replace G by a quotient G' as explained in the footnote of definition 2.1). Hence Th. 2.6 applies in this case, and we have a new family of unitary modular categories. It will be interesting to study these modular categories. As a special case, let $N = 2$. The finite subgroups of $SU(2)$ are classified into $A - D - E$ series. Consider $\mathcal{A}_{SU(2)_k}$ and $G/\mathbb{Z}_2 = \mathbb{Z}_k$, where \mathbb{Z}_2 is the center of $SU(2)$. Note that the coset

$$U(1)_{2k} \subset SU(2)_k$$

has been studied in §3.5 of [X5]. We claim that

$$\mathcal{A}_{SU(2)_k}^G = \mathcal{A}_{U(1)_{2k}} \otimes \mathcal{A}_{SU(2)_k/U(1)_{2k}}.$$

Note that $G \subset U(1)$, and by definition we have

$$\mathcal{A}_{U(1)_{2k}}(I) \otimes \mathcal{A}_{SU(2)_k/U(1)_{2k}}(I) \subset \mathcal{A}_{SU(2)_k}^G(I).$$

But by Lemma 2.2 and §3.5 of [X5],

$$\mu_{\mathcal{A}_{SU(2)_k}} |k|^2 = \mu_{\mathcal{A}_{U(1)_{2k}}} \mu_{\mathcal{A}_{SU(2)_k/U(1)_{2k}}}.$$

It follows by Prop. 2.4 that the inclusion

$$\mathcal{A}_{U(1)_{2k}}(I) \otimes \mathcal{A}_{SU(2)_k/U(1)_{2k}}(I) \subset \mathcal{A}_{SU(2)_k}^G(I)$$

has index 1 which shows that

$$\mathcal{A}_{SU(2)_k}^G = \mathcal{A}_{U(1)_{2k}} \otimes \mathcal{A}_{SU(2)_k/U(1)_{2k}}.$$

Thus the unitary modular categories associated with $\mathcal{A}_{SU(2)_k}^G$ and the corresponding 3-manifold invariants are determined by §3.5 of [X5].

Finally, let us mention that permutation orbifolds (cf. [BDM] and references therein) provide another interesting class of orbifolds. Let us formulate these orbifolds in our setting. Let \mathcal{A} be an irreducible conformal precosheaf. Then \mathcal{A} tensor product itself n times $\mathcal{A}^{\otimes n} := \mathcal{A} \otimes \mathcal{A} \otimes \dots \otimes \mathcal{A}$ is also an irreducible conformal precosheaf. Let $G \subset S_n$ be a finite subgroup of S_n , the permutation group on n letters. Note that any finite group is embedded in a permutation group by Cayley's theorem. There is an obvious action of G on $\mathcal{A}^{\otimes n}$ by permuting the n tensors, and one checks directly by definitions that this action of G on $\mathcal{A}^{\otimes n}$ is proper as defined in §2.3. Note that if \mathcal{A} is μ -rational, so is $\mathcal{A}^{\otimes n}$ by definition. So if \mathcal{A} is μ -rational, by Th. 2.6 we obtain a family of new unitary modular categories from the orbifold $(\mathcal{A}^{\otimes n})^G$. Consider a simple example where $\mathcal{A} = \mathcal{A}_{SU(2)_1}$ and $n = 2$.

Let $G = \mathbb{Z}_2 = S_2$. Then by definition one has $\mathcal{A}_{SU(2)_1} \otimes \mathcal{A}_{SU(2)_1} = \mathcal{A}_{SU(2)_1 \times SU(2)_1}$, and

$$\mathcal{A}_{SU(2)_2}(I) \otimes \mathcal{A}_{SU(2)_1 \times SU(2)_1 / SU(2)_2}(I) \subset (\mathcal{A}^{\otimes^2})^G(I).$$

By computing the μ -index from §3 of [X3] we have

$$\mu_{\mathcal{A}_{SU(2)_2}} \mu_{\mathcal{A}_{SU(2)_1 \times SU(2)_1 / SU(2)_2}} = 4\mu_{\mathcal{A}_{SU(2)_1}}^2$$

and it follows from Prop. 2.4 as in the previous paragraph that

$$\mathcal{A}_{SU(2)_2} \otimes \mathcal{A}_{SU(2)_1 \times SU(2)_1 / SU(2)_2} = (\mathcal{A}^{\otimes^2})^G.$$

The modular category associated with $(\mathcal{A}^{\otimes^2})^G$ and the corresponding 3-manifold invariants are therefore determined by Prop. 3.6.3 of [X5].

In general, the modular matrices of the unitary modular categories associated with the permutation orbifolds above have been written down based on heuristic physics arguments in [Ba]. It will be interesting to do the computations in our framework as in §3 and compare with the results of [Ba].

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